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LETTER TO THE EDITOR

Symmetry properties of one- and two-dimensional Fokker-Planck equations

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Abstract. We study symmetry properties of some Fokker-Planck (FP) equations. In the one-dimensional case, when symmetry groups turn out to be six-parameter ones, this allows us to find changes of variables to reduce such FP equations to the one-dimensional heat equation. We present seven well known FP equations which are reduced to the heat equation. We also study the symmetry and obtain some exact solutions of the Kramers equation.

The one-dimensional FP equation has the form (see, for example, [1])

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} [A(x, t)u] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [B(x, t)u] \tag{1}$$

where $u = u(x, t)$ is the probability density; A and B are differentiable functions. This is the basic equation in the theory of continuous Markovian processes. The following FP equations are of special interest [1, 2]:

(a) diffusion in a gravitational field

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} (gu) + \frac{1}{2} D \frac{\partial^2 u}{\partial x^2} \tag{2}$$

(b) the Ornstein-Uhlenbeck process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} (kxu) + \frac{1}{2} D \frac{\partial^2 u}{\partial x^2} \tag{3}$$

(c) the Rayleigh-type process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[\left(\gamma x - \frac{\mu}{x} \right) u \right] + \frac{1}{2} \mu \frac{\partial^2 u}{\partial x^2} \tag{4}$$

(d) models in population genetics [2]

$$\frac{\partial u}{\partial t} = \frac{\alpha}{2} \frac{\partial^2}{\partial x^2} [(x-c)^2 u] + \beta \frac{\partial}{\partial x} [(x-c)u] \tag{5}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} [(1-x^2)^2 u] \tag{6}$$

$$\frac{\partial u}{\partial t} = \frac{\alpha}{2} \frac{\partial^2}{\partial x^2} [x^2(1-x^2)^2 u] \tag{7}$$

(e) the Rayleigh process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[\left(\gamma x - \frac{\mu}{x} \right) u \right] + \mu \frac{\partial^2 u}{\partial x^2} \tag{8}$$

where $D, g, k, \gamma, \mu, \alpha, \beta, c$ are arbitrary constants.

Using Lie's method [3] one can make sure that the maximal invariance group of equations (2)-(7) is a six-parameter one. The same dimension has the invariance group of the heat equation. It is to be pointed out that these six-parameter groups are different, but they are locally isomorphic. That is why one can reduce equations (2)-(7) to the heat equation.

Theorem 1. The change of variables

$$u(x, t) = f(x, t)w(y(x, t), \tau(x, t)) \tag{9}$$

where the function f and new independent variables y and τ are as follows:

$$f = \exp \left\{ -\frac{g}{D}x - \frac{g^2}{2D}t \right\} \quad y = x \quad \tau = \frac{D}{2}t \tag{10}$$

$$f = \exp\{kt\} \quad y = \exp\{kt\}x \quad \tau = \frac{D}{4k} \exp\{2kt\} \tag{11}$$

$$f = \exp\{2\gamma t\}x \quad y = \exp\{\gamma t\}x \quad \tau = \frac{\mu}{4\gamma} \exp\{2\gamma t\} \tag{12}$$

$$f = \exp \left\{ -\left(\frac{\beta^2}{2\alpha} + \frac{\beta}{2} + \frac{\alpha}{8} \right) t \right\} (x - c)^{-(3/2 + \beta/\alpha)} \quad y = \sqrt{2/\alpha} \ln(x - c) \quad \tau = t \tag{13}$$

$$f = \exp\{-t\}(1 - x^2)^{-3/2} \quad y = \frac{1}{2} \ln \frac{1+x}{1-x} \quad \tau = t \tag{14}$$

$$f = \exp \left\{ -\frac{\alpha}{8} t \right\} x^{-3/2}(1 - x)^{-3/2} \quad y = \ln \frac{x}{1-x} \quad \tau = \frac{\alpha}{2} t \tag{15}$$

reduce equations (2)-(7), correspondingly, to the heat equation

$$w_\tau = w_{yy}. \tag{16}$$

The proof can be easily obtained by inspection.

Remark 1. One can prove a more general statement. Equation (1) with coefficients

$$A(x, t) = A(x) \quad B(x, t) = B = \text{constant} \tag{17}$$

is reduced to the heat equation if and only if

$$A^2 + B \frac{\partial A}{\partial x} = c_2 x^2 + c_1 x + c_0 \tag{18}$$

where c_0, c_1, c_2 are arbitrary constants. Note that equations (2)-(4) satisfy condition (18) and equation (8) does not. The general solution of equation (18), which is a Riccati one, cannot be obtained in quadrature [4].

Remark 2. The FP equation of the form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} [(a(t)x + b(t))u] + c(t) \frac{\partial^2 u}{\partial x^2} \tag{19}$$

was considered in [5, 6] and by means of algebraic methods a class of solutions of it was obtained. This result can be easily obtained if we note that equation (19) is reduced to the heat equation (16) by the substitution (9) with

$$\begin{aligned} f &= \exp\left\{-\int_0^t a(s) ds\right\} \\ y &= \exp\left\{-\int_0^t a(s) ds\right\}x - \int_0^t b(s) \exp\left\{-\int_0^s a(\xi) d\xi\right\} ds \\ \tau &= \int_0^t c(s) \exp\left\{-2\int_0^s a(\xi) d\xi\right\} ds. \end{aligned} \tag{20}$$

Now consider the two-dimensional FP equation which describes the motion of a particle in a fluctuating medium (so-called Brownian movement)

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} (yu) + \frac{\partial}{\partial y} (V'(x)u) + \gamma \frac{\partial}{\partial y} \left(yu + \frac{\partial u}{\partial y}\right) \tag{21}$$

where $u = u(t, x, y)$, γ is a constant and $V(x)$ is the potential (its gradient defines the exterior force). Equation (21) is known as Kramers equation [1].

Theorem 2. The maximal invariance group of the free Kramers equation

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} (yu) + \gamma \frac{\partial}{\partial y} \left(yu + \frac{\partial u}{\partial y}\right) \tag{22}$$

is a six-dimensional Lie group generated by the following operators:

$$\begin{aligned} P_0 &= \partial_t & P_1 &= \partial_x & I \\ G_1 &= t\partial_x + \partial_y + \frac{1}{2}(y + \gamma x) \\ S_1 &= e^{\gamma t} \left(\frac{1}{\gamma} \partial_x + \partial_y + y\right) & T_1 &= e^{-\gamma t} \left(\frac{1}{\gamma} \partial_x - \partial_y\right) \end{aligned} \tag{23}$$

which satisfy the commutation relations

$$\begin{aligned} [P_0, G_1] &= P_1 & [P_0, S_1] &= \gamma S_1 \\ [P_0, T_1] &= -\gamma T_1 & [P_1, G_1] &= -\frac{1}{2}\gamma & [T_1, S_1] &= I \end{aligned} \tag{24}$$

(the rest of the commutators are equal to zero).

The proof can be obtained by Lie's method.

Remark 3. One can prove a more general statement: the widest symmetry group of equation (21) is achieved when $V'(x) = c_1x + c$ (c_1, c are arbitrary constants) and it is a six-parameter group.

Remark 4. The change of variables

$$u = w(\tau, \xi, \eta) \quad \tau = t \quad \xi = x - (c/\gamma)t \quad \eta = y - (c/\gamma) \quad (25)$$

reduces equation (21) with $V'(x) = c$ to the free Kramers equation (22).

Let us write down the final transformations generated by operators (23). Operators P_0 and P_1 generate translations on variables t and x ; I generates the identical transformation; G_1 generates

$$\begin{aligned} t' &= t & x' &= x + at & y' &= y + a \\ u'(x') &= \exp\left\{-\frac{1}{2}[ay + (a^2/2)(1 + \gamma t) + \gamma ax]\right\}u(x) \end{aligned} \quad (26)$$

S_1 generates

$$\begin{aligned} t' &= t & x' &= x + (b/\gamma) e^{\gamma t} & y' &= y + b e^{\gamma t} \\ u'(x') &= \exp\{by e^{\gamma t} - (b^2/2) e^{2\gamma t}\}u(x) \end{aligned} \quad (27)$$

T_1 generates

$$\begin{aligned} t' &= t & x' &= x + (\theta/\gamma) e^{-\gamma t} & y' &= y - \theta e^{-\gamma t} \\ u'(x') &= u(x) \end{aligned} \quad (28)$$

where a, b, θ are group parameters. It is appropriate to write here the corresponding formulae of generating solutions which follow from (26)-(28) (the general theory is contained in [7]):

$$u_{II}(t, x, y) = \exp\left\{\frac{a}{2}\left[y + \frac{a}{2}(1 + \gamma t) + \gamma x\right]\right\}u_I(t', x', y') \quad (29)$$

$$u_{II}(t, x, y) = \exp\left\{-by e^{\gamma t} + \frac{b^2}{2} e^{2\gamma t}\right\}u_I(t', x', y') \quad (30)$$

$$u_{II}(t, x, y) = u_I(t', x', y') \quad (31)$$

where t', x', y' are given in (26)-(28) respectively.

It will be noted that transformations (26) are just the Galilean ones as soon as the variable y in the Kramers equation is taken to be velocity of the particle.

A well known solution of the Kramers equation (21) is the Boltzmann distribution

$$u(x, y) = \mathcal{N} \exp\{-v(x) - \frac{1}{2}y^2\} \quad (32)$$

(\mathcal{N} is a normalisation constant). It is a stationary solution. Applying this to (32) with $V = 0$, formulae (29)-(31), one can easily obtain a non-stationary solution of equation (22).

According to the algorithm of [8, 7] and using the operator from (23) we find the ansatz

$$u(t, x, y) = \exp\left\{-\frac{y^2}{2}\right\}\varphi(\omega_1, \omega_2) \quad \omega_1 = t, \omega_2 = \gamma x - y. \quad (33)$$

Substitution of (33) into (22) gives rise to the heat equation

$$\frac{\partial \varphi}{\partial \omega_1} - \gamma \frac{\partial^2 \varphi}{\partial \omega_2^2} = 0. \quad (34)$$

The simplest solution of (34) is $\varphi = \text{constant}$, but it is the solution that leads, together with the ansatz (33), to the Boltzmann distribution (32). It is clear that by using solutions of the heat equation (34) and the ansatz (33) one can construct many partial solutions of equation (22). For example, the fundamental solution of (34) and (33) results in the following solution of equation (22):

$$u(t, x, y) = \frac{1}{\sqrt{4\pi\gamma t}} \exp\left\{-\frac{y^2}{2} - \frac{(\gamma x - y)^2}{4\gamma t}\right\}. \quad (35)$$

The operator T_1 from (23) leads to the ansatz

$$u = \tilde{\varphi}(\omega_1, \omega_2) \quad \omega_1 = t \quad \omega_2 = \gamma x + y \quad (36)$$

which reduces (23) to the heat equation (34), where $\varphi = e^{\gamma\omega_1} \tilde{\varphi}(\omega_1, \omega_2)$.

A great number of partial solutions of equation (23) can be found by means of the method described in [9] (see also [7]). So, if $u_0 = u_0(t, x, y)$ is a solution of the equation, then the functions Qu_0, Q^2u_0, \dots will also be solutions with any symmetry operator Q .

For example, starting from $u_0 = e^{\gamma t}$, we find

$$\begin{aligned} u_1 &= 2G_1 e^{\gamma t} = e^{\gamma t}(\gamma x + y) \\ u_2 &= G_1 u_1 = e^{\gamma t}[(\gamma t + 1) + \frac{1}{2}(\gamma x + y)^2] \\ &\dots \end{aligned} \quad (37)$$

Analogously, by means of the operator T_1 from (23), we find, starting from $u_0 = \exp\{-y^2/2\}$,

$$\begin{aligned} u_1 &= T_1 \exp\left\{-\frac{y^2}{2}\right\} = y \exp\left\{-\left(\gamma t + \frac{y^2}{2}\right)\right\} \\ u_2 &= T_1 u_1 = (y^2 - 1) \exp\left\{-\left(2\gamma t + \frac{y^2}{2}\right)\right\} \dots \end{aligned} \quad (38)$$

Solutions (35), (37) and (38) can be multiplied by the formulae of the generating solutions (29)-(31).

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